p-1

Construction of TMF

- Survey of EC: phrase existence of $(\mathcal{M}_{ell}, \mathcal{O}_{top})$ as a moduli problem. Elliptic curves over E_∞-rings. Lurie's representability ⇒ derived Deligne-Mumford stack. Must show π_0 of this is $\cong \mathcal{M}_{ell}$.

- GHM approach. Start with $\overline{\mathcal{M}}_{ell}$, work with presheaves of E_∞ rings thereon. TMF is supposed to exist integrally; we'll construct it prime-by-prime and rationally. Just as we have

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \prod_p \mathbb{Z}_p \arrow[d] \\ \mathbb{Q} \arrow[r] & \prod_p \mathbb{Q}_p \end{tikzcd}$$

a fun way to define $\mathbb{Z}$, we'll have a ~~pod~~ pb square

$$\begin{tikzcd} \mathcal{O}^{top} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \prod_p (i_p)_* \mathcal{O}^{top}_p \arrow[d, "obv"] \\ (i_{\mathbb{Q}})_* \mathcal{O}^{top}_{\mathbb{Q}} \arrow[r, "?"'] & \left(\prod_p (i_p)_* \mathcal{O}^{top}_p\right)_{\mathbb{Q}} \end{tikzcd}$$

$i_p, i_{\mathbb{Q}}$. Must describe all obj's and bottom map.

Recall the "chromatic square"

$$\begin{tikzcd} L_{E(n)} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & L_{K(n)} X \arrow[d] \\ L_{E(n-1)} X \arrow[r] & L_{E(n-1)} L_{K(n)} X \end{tikzcd}$$

similar in spirit to the above. Here n will be 2,

$\mathcal{O}_{top}$ should be $E(2)$-local, and everything is p-complete so $E(1)$-localization $= K(1)$-loc. Motivated by this, we'll define a pb of sheaves of

$$\begin{tikzcd} \mathcal{O}^{top}_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & (\iota_{ss})_* \mathcal{O}^{top}_{K(2)} \arrow[d] \\ (\iota_{ord})_* \mathcal{O}^{top}_{K(1)} \arrow[r, "?"'] & ((\iota_{ss})_* \mathcal{O}^{top}_{K(2)})_{K(1)} \end{tikzcd}$$

Again, must define all objs and bottom map.

Note: $\mathcal{M}^{ss}_{ell}$ is given by completion at ~~a~~ the closed sub stack

$$(\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p} \to (\overline{\mathcal{M}}_{ell})$$

and $\mathcal{M}^{ord}_{ell}$ is the moduli stack of ell curves over p-complete rings with ord reduction. Both

$$\mathcal{M}^{ss}_{ell} \text{ and } \mathcal{M}^{ord}_{ell} \text{ map to } (\overline{\mathcal{M}}_{ell})_p.$$

To construct $\mathcal{O}^{top}_{K(2)}$: let $f : R \to \mathcal{M}^{ss}_{ell}$ be a formal étale open. Formal étaleness means exactly that f is the universal deformation of $f_0 : R/I \to (\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p}$. Plus, $(\mathcal{M}^{ss}_{ell})_{\mathbb{F}_p}$ is 0-dim, so $R/I \cong \prod_i \mathbb{F}_{q_i}$

Serre-Tate: to deform an EC, enough to deform its p-div group. Since f is ss, this $=$ formal group. We know the univ deformation of a formal group of ht 2 over $\mathbb{F}_q$: It's over

$$\mathbb{Z}_q[[u_1]].$$

Hopkins-Miller: we can spectrify this. ~~The value~~

$\Rightarrow \mathcal{O}^{top}_{K(2)}$.

Let $\mathcal{M}^{ord}_{ell}(p^\infty)$ be the ~~object representing~~ moduli object of (C, η), where C is an ell curve over a p-complete ring and $\eta : \hat{C} \xrightarrow{\sim} \mu_{p^\infty}$. Obv $\mathcal{M}^{ord}_{ell}(p^\infty) \to \mathcal{M}^{ord}_{ell}$.

What kind of obj is $\mathcal{M}^{ord}_{ell}(p^\infty)$? Actually (hard work) it's a formal affine scheme. "Coordinatising" the fg is enough to trivialize ω, so if

$$\mathcal{M}^{ord}_{ell}(p^\infty) = \mathrm{Spf}(\hat{V_\infty})$$

we call $\hat{V_\infty}$ the ring of p-adic modular functions.

<u>Prop.</u> $\hat{V_\infty}$ is a θ-ring.

We need ring homos ψ^k for $k \in \mathbb{Z}_p^\times$, ψ^p lifting Frob, a ring homo, and θ with $\psi^p(x) = x^p + p\theta(x)$. $\hat{V_\infty}$ is torsion-free, so we don't need to worry about θ.

$\mathbb{Z}_p^\times$ acts on μ_{p^∞}, so ψ^k comes from changes of co-ordinates. ψ^p comes from studying the pth power endomorphism of a curve: It is given on q-expansions by ~~raising~~ $q \mapsto q^p$.

To construct $\mathcal{O}^{top}_{K(1)}$, we first construct its spectrum of sections over $\mathcal{M}^{ord}_{ell}(p)$, (which is also formally affine.)

Let R be a p-complete ring.

Thm. If W is defined by

$$\begin{tikzcd} \mathrm{Spf}(W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[d] \\ \mathrm{Spf}(R) \arrow[r] & \mathcal{M}_{ell}^{ord} \end{tikzcd}$$

then $W \cong (K_p^\wedge)_0 E$ (elliptic se). Moreover, this iso entwines the Adams ops with the $\mathbb{Z}_p^\times$ action on the right.

Pf Comes from the fact that

$$\begin{tikzcd} \mathcal{M}_{ell}^{ord}(p^\infty) \arrow[r] \arrow[d] & \mathrm{Spf}(\pi_0 K_p^\wedge) \arrow[d] \\ \mathcal{M}_{ell}^{ord} \arrow[r] & \mathcal{M}_{fg} \end{tikzcd}$$

~~We apply this to~~ $\cancel{\mathrm{Spf}(R) = \mathcal{M}_{ell}^{ord}(p)}$.

nb. $(K_p^\wedge)_0 E$ is a θ-ring, but $W \cong (K_p^\wedge)_0 E$ may not preserve ψ^p. If it doesn't, you don't have the right E_∞ st.

We apply this to $\mathrm{Spf}(R) = \mathcal{M}_{ell}^{ord}(p)$. The obstructions to realizing W_* as $(K_p^\wedge)_*$ of a $K(1)$-local E_∞-ring vanish. Let this E_∞-ring be denoted $tmf(p)^{ord}$.

$$W_{2*} = \omega^{\otimes *}(\mathrm{Spf}(W))$$

Define $tmf_{K(1)} = (tmf(p)^{ord})^{h(\mathbb{Z}/p)^\times}$.

Let $\mathrm{Spf}(R) \xrightarrow{f} \mathcal{M}_{ell}^{ord}$ be étale. Then the θ-ring

$$W \cong (K_p^\wedge)_* E_f$$

is a $K^\wedge_{p*}\mathrm{tmf}_{K(1)}$-algebra. There are θ-ring homology groups containing obstructions to realizing it as a $\mathrm{tmf}_{K(1)}$-alg, and these too vanish. This provides a construction of $\mathcal{O}^{top}_{K(1)}$. The construction of

$$\alpha : (\iota_{ord})_* \mathcal{O}^{top}_{K(1)} \longrightarrow ((\iota_{ss})_* \mathcal{O}^{top}_{K(2)})_{K(1)}$$

is similar, in spirit; I won't go into details.

We have constructed $\mathcal{O}^{top}_p$. Now let's construct $\mathcal{O}^{top}_{\mathbb{Q}}$. The whole E_∞ game collapses rationally, and so this is much easier. If

$$f : \mathrm{Spec}(R) \to (\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$$

is étale, then the sections over f should be

$$\pi_{2t} E_f = \Gamma((f^*\omega)^{\otimes t})$$

Let R_* be defined by $R_{2t} = \Gamma((f^*\omega)^{\otimes t})$. Then we define

$$\mathcal{O}^{top}_{\mathbb{Q}}(f) = H(R_*).$$